Concorso per l'ammissione al corso di Dottorato di Ricerca in Matematica, Informatica, Statistica - Ciclo XXX

Curriculum in Mathematics

Examination 2

Applicants are required to do some of the following exercises

1. Let $S(n, \mathbb{R})$ be the space of real $n \times n$ symmetric matrices with real coefficients.

(a) Prove that the map $q : S(n, \mathbb{R}) \times S(n, \mathbb{R}) \to \mathbb{R}$ given by

$$q(A, B) = \mathrm{Tr}(A \cdot B), \qquad A, B \in S(n, \mathbb{R})$$

is a bilinear form which is positive definite.

(b) Given $A \in S(n, \mathbb{R})$, prove that the map $L_A(X) := AXA$ defines a linear endomorphism of $S(n, \mathbb{R})$.

(c) Prove that L_A is selfadjoint w.r.t. q and that it is diagonalizable for every $A \in S(n, \mathbb{R})$.

2. Let R be a ring such that, for every $x \in R$, we have $x^2 = x$ (such a ring is said to be a *boolean ring*).

(a) Show that the ring R has characteristic 2 and that R is commutative.

(b) Given $x, y \in R$ show that the ideal they generate (x, y) is principal. Use this fact to show that every finitely generated ideal of R is principal. Do there exist boolean rings in which not all the ideals are principal?

3. Let G be a group and set $S = \{x^2 y^{-2} \mid x, y \in G \setminus \{1\}\}$.

(a) Prove that $N = \langle S \rangle$ is a normal subgroup of G.

(b) Let $H = G/N$. Show that, for every $u, v \in H \setminus \{1\}$, we have $u^2 = v^2$. Use this fact to prove that, for every $h \in H$, one has $h^2 = 1$.

4. Let $k \in (0, +\infty)$. Prove that

(a) if $u \in C^1(\mathbb{R})$ satisfies $u'(x) \geq k\, u(x)$ for all $x \in \mathbb{R}$, then

$$u(x)e^{-kx} \geq u(y)e^{-ky} \qquad \forall\, x,\, y \in \mathbb{R},\ x \geq y;$$

(b) if $u \in C^2(\mathbb{R})$ satisfies $u''(x) \geq k^2\, u(x)$ for all $x \in \mathbb{R}$ and $\sup_{\mathbb{R}} u < +\infty$, then

$$u(x)e^{kx} \leq u(y)\, e^{ky} \qquad \forall\, x,\, y \in \mathbb{R},\ x \geq y.$$

5. Let $u : \mathbb{R} \mapsto \mathbb{R}$ be a convex function. Prove that

(a) if there are points $\alpha < \beta < \gamma$ such that $u(\alpha) = u(\beta) = u(\gamma)$, then u is constant on the interval $[\alpha, \gamma]$;

(b) if u is bounded from above, then u is constant on the whole $\mathbb{R}$;

(c) either u is constant or at least one among the limits of u at $\pm\infty$ is $+\infty$.

6. Introduce the Jacobi iterative method to solve a linear system $A\mathbf{x} = \mathbf{b}$, $A \in \mathbb{R}^{n\times n}, \mathbf{b} \in \mathbb{R}^n$. Then, for A and $\mathbf{b}$ selected as follows,

$$A = \begin{pmatrix} \alpha & -2 \\ 1 & 1-\alpha \end{pmatrix}, \quad \mathbf{b} = (-1\,, -3)^T\,,$$

precise for which values of $\alpha \in \mathbb{R}$ the method can't be used and if it is convergent when $\alpha = 3$ (justify the answer).

Finally, selecting $\alpha = 3$, execute two iterations of the method, using $\mathbf{x}^{(0)}$ as starting vector, with $\mathbf{x}^{(0)} = (2\,, 1)^T$, and compare the euclidean norm of the initial residual vector with that of the final one.

7. Explain when an eigenvalue of a square matrix is said dominant and prove that it exists for the following matrix,

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 8 \end{pmatrix}.$$

Chosen a non vanishing starting vector $\mathbf{x}^{(0)} \in \mathbb{R}^3$, analyse qualitatively the behavior of the vector sequence defined as follows,

$$\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}\,, \quad k = 0, 1, \dots\,,$$

Explain in particular if it can be used to approximate the dominant eigenvalue of A and how. Finally, underline which computational problem arises at a certain iteration when a finite arithmetic is used for the computation of the above sequence.

8. Give the definition of covering space of a topological space.

(a) Let X, Y be two Hausdorff topological spaces and $p : X \to Y$ be a local homeomorphism which is surjective and with finite fibre. Is p a covering map ?

(b) Let $\mathbb{S}^2$ be the 2-sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = 1\}$ and let $X := \mathbb{S}^2 \cup \{(x, y, z) \in \mathbb{R}^3 |\ x = y = 0,\ -1 \le z \le 1\}$. Can X cover $\mathbb{S}^2$ or be covered by $\mathbb{S}^2$?

(c) Give a description of the universal covering $\pi : \tilde{X} \to X$ using the following steps:

(c1) describe $\pi^{-1}(\mathbb{S}^2)$;

(c2) if $\gamma := X \cap \{x = 0,\ y \geq 0\}$, find $\pi^{-1}(\gamma)$.

9. Let X be a topological space and consider the following relation: if $p, q \in X$, then $p \sim q$ if and only if $d(p) = d(q)$ for every continuous map $d : X \to D$ where D is any discrete topological space.

(a) Prove that $\sim$ is an equivalence relation;

(b) Prove that the equivalence classes of $\sim$ are closed subspaces of X, which we will call *quasi-components*;

(c) Prove that every connected component of X is contained in a quasi-component;

(d) Let X be the topological subspace of $\mathbb{R}^2$ given by $X = \left(\bigcup_{n\in\mathbb{N}, n>0}\{\frac{1}{n}\} \times [0, 1]\right) \cup \{(0, 0)\} \cup \{(0, 1)\}$. Find all the connected components and the quasi-components of X.

10. Given an urn with unknown composition U, containing 4 black and white balls, not all of the same color, let E be the event "from the urn U a white ball is drawn".

Consider now the experiment: a regular 6-sided dice is rolled and let k be the (random) result, then a ball is drawn from the urn U, its color is noted, and it is returned to the urn a number k of times. Let F be the event "in the k draws exactly 4 white balls are obtained" and H_i the event "the urn U contains i white balls" with $i = 1, 2$.

(a) Say which probabilistic information about the composition of the urn can be deduced from the assumption $P(E) = \frac{1}{3}$.
Explain way, under this assumption, it is not possible to assume that all the possible compositions are equally probable.

(b) After mentioning the Bayes theorem, show if the assumption made in a) is compatible with the following: $P(H_2|F) = \frac{1}{3}$. In the affirmative case compute the probability of H_1, under both the assumptions.

11. Let X be a random variable with density function

$$f(x) = \begin{cases} 0 & x < -1 \\ ax^3 & -1 \leq x < 0 \\ \frac{x+a^2}{2} & 0 \leq x < 1 \\ 0 & x \geq 1 \end{cases}$$

(a) After recalling the properties of a density function find the values of parameter a such that f is a density function and then compute the corresponding cumulative distribution function. Compute also the expected value and the variance of the variable $Y = 7 - 3X$

(b) Recall the properties of the Pearson (product-moment) correlation coefficient $\rho(.,.)$ and compute $\rho(X, Y)$.

(c) Say if the variable $Z = X^2 + 1$ has density and in the case of an affirmative answer compute it.

12. On a horizontal plane, a material point P_1 of mass m_1, is constrained to move on a circle of radius R whose center can be taken as the origin of the Cartesian reference frame. On the point P_1 acts only a force $\vec{F}(x, y) = -a\left(y^2\vec{\imath} + 2xy\vec{\jmath}\right)$, where $a > 0$ is a dimensional constant, $\vec{\imath}, \vec{\jmath}$ are the unit vectors of the Cartesian reference (figure 1) and x, y are the coordinates of the generic point of the plane. We ask

(a) to prove that $\vec{F}$ is conservative and to write the potential energy.

Chosen as the Lagrangian coordinate for P_1 the angle ϑ shown in the figure, we ask

(b) to identify the equilibrium configurations of P_1 and to discuss their stability.

Along with the point P_1, now consider a point P_2, of mass m_2, on which does not act any force, and which moves keeping its ordinate y_2 constantly equal to that of P_1. We ask

(c) to identify the number of degrees of freedom of the system, the suitable Lagrangian coordinate, and to write the corresponding Lagrange equations of the second kind.

13. A material point P of mass m moves without friction on a horizontal line. A rigid rod of negligible mass and length ℓ has one end point in P. A material point Q of mass m is fixed to the other end of the rod. On the system only acts the weight force, directed vertically.

(a) Write the Lagrangian for the system and the equations of motion.

(b) Determine the equilibrium solutions and discuss their stability. Moreover, calculate the frequency of small oscillations close to possibly stable equilibria.

(c) Determine the first integrals of motion (if any).

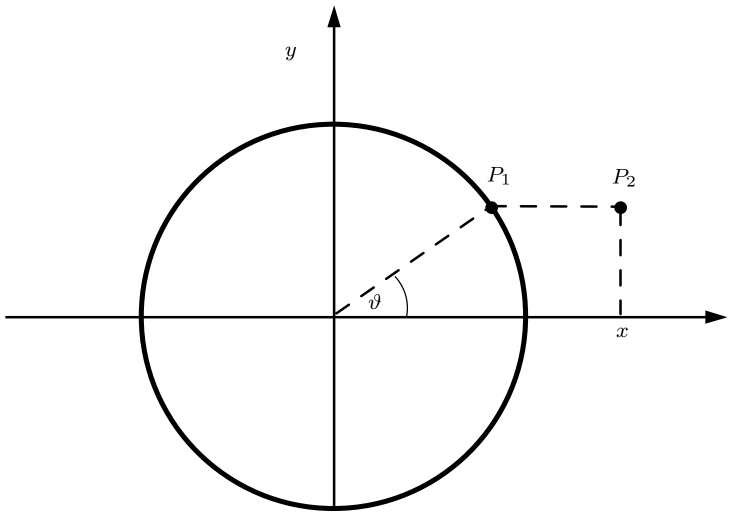

Figure 1.