Prova di ammissione al Corso di Dottorato di Ricerca in Matematica, Informatica e Statistica - Ciclo XXXII

Curriculum in Mathematics

Text 3

Exercise 1. Let A be an abelian group and $n \in \mathbb{N}$ any natural number. Prove that $\sigma_n : A \longrightarrow A$ defined by $\sigma_n(a) = a^n$ is a group homomorphism. Show that if σ_n is surjective for all $n > 1$, then A has no maximal subgroups. If T indicates the subgroup of elements of A of finite period, prove that $\sigma_n(T) \leq T$ for all n. Use this fact to prove that, setting $\phi_n(a+T) = \sigma_n(a)+T$, defines an endomorphism ϕ_n of A/T. Find $\ker(\phi_n)$. Prove that $B = \bigcup_{0 \neq n \in \mathbb{N}} \ker(\sigma_n)$ is a subgroup of A.

Exercise 2. Let G be a group and, for each $x, y \in G$, set $[x, y] = x^{-1}y^{-1}xy$.

1. Prove that the subgroup $D = \langle [x, y] \mid x, y \in G \rangle$ is normal in G.
2. If $\sigma : G \longrightarrow A$ is a group homomorphism and A is abelian, show that $D \leq \ker(\sigma)$.
3. Show that $D = 1$ if and only if G is abelian.
4. If $\mathcal{A} = \{N \trianglelefteq G \mid G/N \text{ is abelian } \}$, prove that $D = \cap_{N \in \mathcal{A}} N$.

Exercise 3. Consider the Cauchy problem:

$$\begin{cases} y'(x) = y^3(x) - 3y(x) + 1, \\ \\ y(1) = 0. \end{cases}$$

1. Prove that it admits a solution defined in $\mathbf{R}$.
2. Draw the graph of the solution (including monotonicity and concavity).

Exercise 4. Let f be a real-valued function defined in $\mathbf{R}^n$, 1-homogeneous:

$$f(\lambda x) = \lambda\, f(x) \quad \forall\, x \in \mathbf{R}^n,\, \forall\, \lambda \geq 0.$$

Assume moreover that f is sub-additive:

$$f(x + y) \leq f(x) + f(y) \quad \forall\, x, y \in \mathbf{R}^n.$$

1. Prove that f is convex in $\mathbf{R}^n$.
2. Let:
$$K = \{x \in \mathbf{R}^n \colon (x, y) \leq f(y) \text{ for every } y \in \mathbf{R}^n\},$$
where $(\cdot, \cdot)$ denotes the standard scalar product in $\mathbf{R}^n$; prove that:

(a) K is convex;

(b) K is closed;

(c) K is bounded.

Exercise 5. Let $n \in \mathbf{N} - \{0\}$ and let K be a field. Denote the set of the matrices $n \times n$ with entries in K by $M(n \times n, K)$. Let $A, B \in M(n \times n, K)$.

Prove that $\begin{pmatrix} I & B \\ A & I \end{pmatrix}$ is invertible if and only if $det(I - AB) \neq 0$ and $det(I - BA) \neq 0$ (where det denotes the determinant and I denotes the identity matrix $n \times n$).

Exercise 6. Let $S^1 = \{x \in \mathbf{R}^2 | \, |x| = 1\}$ and let T be the torus $S^1 \times S^1$. Let C be the following subset of T: $\{(0, 1)\} \times S^1$. Let P and Q be two distinct points of $T - C$.

Denote by X the topological space obtained from T by attaching a 2-cell B^2 along C (that is, identify the border of B^2 with C) and a 1-cell B^1 along $P \cup Q$ (that is, identify the border of B^1 with $P \cup Q$).

Let R be a ring. Give the definition of first fundamental group and calculate the first fundamental group and the singular homology modules over R of X.

Exercise 7. A homogeneous rigid material system consists of a rod OA of length l and mass m_1 and of a circumference arc BC, whose amplitude is 2α and whose mass is m_2. The circumference radius is r. The BC arch is welded in its midpoint to the rod extreme A (see figure 1).

(a). Prove that the axis of the reference system $\{O, x, y, z\}$ shown in figure 1 – left picture (the z axis is perpendicular to the plane Oxy) are principle axis of inertia and compute the relative inertia matrix.

(b). The rigid system is now pivoted (frictionless) at the point O of the vertical plane and the weight force is as in figure 1 – right picture. Determine the small oscillations frequency.

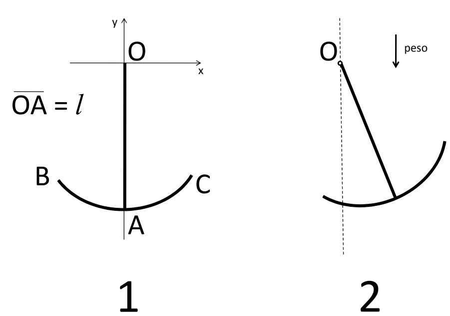

Figure 1:

Exercise 8. A material point P, whose mass is m, can slide along a circle of radius R, placed in a vertical plane, as shown in figure 2. The material point is attracted toward the y-axis by an elastic force whose constant is k (see the figure). All constraints are frictionless and the weight is as in figure 2. Use the angle θ as lagrangian parameter.

(a). Determine equilibrium positions of P and discuss their stability.

(b). Determine the relative equilibrium positions and discuss their stability, in the case in which the circle rotates with constant angular velocity ω around the y axis.

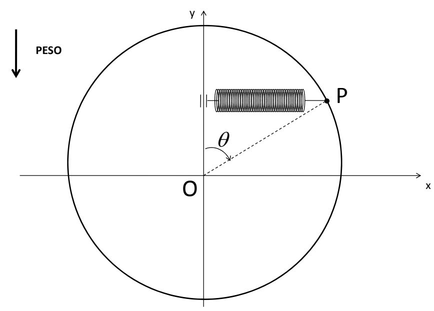

Figure 2:

Exercise 9. A random variable X has probability density function $f(t)$ given by:

$$f(t) := \begin{cases} 0 & \text{if} \quad t \leq -7 \\ ct^2 & \text{if} \quad -7 < t < 1 \\ 2e^{-6(t-1)} & \text{if} \quad t > 1 \end{cases} \tag{1}$$

(a) Compute the constant c that gives rise to a well defined density function.

(b) Compute the cumulative distribution function of X.

(c) Compute the expected value of X.

Exercise 10. Consider two discrete random variables X and Y with joint probability distributions given by the following table:

$P(X=x, Y=y)$		Y: -2	1	3	
X	-2	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	.
	-1	$\frac{1}{8}$	0	$\frac{1}{16}$	.
	0	0	$\frac{1}{16}$	$\frac{2}{16}$	.
	2	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	.
.		.	.	.	

(a) What are the marginal distributions of X and Y? And what is the conditional probability distribution of X given $Y = 3$?

(b) Compute the expected value of X given $Y = 3$ and the variance of X given $Y = 3$.

(c) Compute the covariance $cov(X, Y)$. Are the random variables X, Y independent? Justify your answer.

Exercise 11. Let $f(x) = -|x|x + 9x + 50$, with $x \in \mathbf{R}$. Say what is error committed by the Simpson method applied to compute the integral of the given function in the interval $[-5, 5]$ if the domain is divided into four sub-intervals. If we used 21 sub-intervals would the error decrease?

Exercise 12. Let $f(x) = -|x|x + 9x + 50$, with $x \in \mathbf{R}$. Say if the tangent method, starting from the following initial points, converges to the root of the given function

$$x_0 = -\frac{9}{2},$$

$$x_0 = +5,$$

$$x_0 = -5.$$